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# Directed polymers on a Cayley tree with spatially correlated disorder

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Received 21 January 1998

**Abstract.** In this paper we consider directed walks on a tree with a fixed branching ratio  $K$  at a finite temperature  $T$ . We consider the case where each site (or link) is assigned a random energy uncorrelated in time, but correlated in the transverse direction, i.e. within the shell. In this paper we take the transverse distance to be the hierarchical ultrametric distance, but other possibilities are discussed. We compute the free energy for the case of quenched disorder and show that there is a fundamental difference between the case of short-range spatial correlations of the disorder which behave similarly to the non-correlated case considered previously by Derrida and Spohn and the case of long-range correlations which have a totally different overlap distribution (approaching a single delta function about  $q = 1$  for large  $L$ , where  $L$  is the length of the walk). In the latter case the free energy is not extensive in  $L$  for the intermediate and also relevant range of  $L$  values, although in the true thermodynamic limit extensivity is restored. We identify a crossover temperature which grows with  $L$ , and whenever  $T < T_c(L)$  the system is always in the low-temperature phase. Thus, in the case of long-ranged correlation as opposed to the short-ranged case a phase transition is absent.

## 1. Introduction

The problem of directed polymers in a random medium can be formulated on a lattice [1–5], or in the continuum limit [6–12]. On a lattice there is a random energy associated with each bond (or site). Walks (or polymers) start at a given point and are allowed to proceed only along the positive direction of one of the coordinates which is referred to as ‘time’. The other coordinates are referred to as ‘transverse’. The partition function is given by

$$Z_L(\beta) = \sum_w e^{-\beta E(w)} \quad (1)$$

where the sum is over all walks  $w$  of  $L$  steps and

$$E(w) = \sum_{(ij) \in w} \epsilon_{ij} \quad (2)$$

is the sum of the random energies along the walk.  $\beta = 1/T$  is the the inverse temperature in the proper units.

In the continuum limit the partition function is given by the functional integral

$$Z(\beta) = \int [D\mathbf{x}(t)] \exp \left\{ -\beta \int_0^L dt \left[ \frac{1}{2} \dot{\mathbf{x}}(t)^2 + V(\mathbf{x}(t), t) \right] \right\} \quad (3)$$

where  $\mathbf{x}(t)$  is the  $(d - 1)$ -dimensional transverse position of the polymer at time  $t$  and  $V(\mathbf{x}(t), t)$  is the random potential. The term  $\frac{1}{2} \dot{\mathbf{x}}(t)^2$  measures the bending energy of the polymer.

On the special lattice of a Cayley tree and with uncorrelated disorder

$$\langle \epsilon_{ij} \epsilon_{lm} \rangle = g \delta_{il} \delta_{jm} \quad (4)$$

many properties of the model could be extracted analytically [13, 14], such as the free energy and the probability of overlaps between two walks. The model exhibits a phase transition at finite temperature  $T_c$ . Define  $q(w, w')$  to be the fraction of their length that two walks  $w, w'$  of length  $L$  spend together. The probability distribution for overlaps is then given by

$$P(q) = \left\langle \frac{1}{Z_L^2} \sum_w \sum_{w'} \delta(q - q(w, w')) \exp(-\beta E_w - \beta E_{w'}) \right\rangle. \quad (5)$$

For  $T > T_c$  it was found that the probability of overlaps is a single delta function at  $q = 0$ , whereas for  $T < T_c$  the probability distribution consists of a weighted sum of two delta functions

$$P(q) = \frac{T}{T_c} \delta(q) + \left(1 - \frac{T}{T_c}\right) \delta(q - 1). \quad (6)$$

This distribution implies that the free energy landscape consists of many valleys separated by large barriers. Two walks lying in the same valley have an overlap of  $q = 1$ , whereas walks lying in different valleys have zero overlap.

In the continuum limit the model has been treated for general  $d$  by the variational approximation [8] and by the  $1/d$ -expansion [11], both valid for large  $d$ . For the special case of  $d = 2$  a Bethe ansatz technique yielded some exact results [6]. In the continuum limit the case of both short- and long-ranged correlations of the disorder have been considered. The correlations have been defined by

$$\langle V(\mathbf{x}, t) V(\mathbf{x}', t') \rangle = \delta(t - t') F((\mathbf{x} - \mathbf{x}')^2) \quad (7)$$

and classified as short or long ranged according to the form of  $F(y^2)$ . For the short-ranged case one usually takes [8, 15, 16]

$$F(y^2) = \frac{g}{\gamma - 1} (a_0 + y^2)^{(1-\gamma)} \quad (8)$$

where  $g > 0$  is the strength of the disorder and  $\gamma > 2$  determines the range of the correlations.

The case of  $1 < \gamma < 2$  is considered long ranged. Also considered long ranged are correlations of the form

$$F(y^2) = a_0 - \frac{g}{1 - \gamma} y^{2(1-\gamma)} \quad (9)$$

with  $0 \leq \gamma < 1$ . The constant  $a_0$  is important to maintain the requirement  $|F(y^2)| \leq F(0)$  dictated by the Schwarz inequality for the appropriate  $y$ -range, but is sometimes neglected in the literature [8], more than likely because it contributes only a trivial constant to the free energy. In the case of a correlation of the form (9),  $a_0$  has to be very big, such that  $[a_0(1 - \gamma)/g]^{1/(2(1-\gamma))}$  is greater than the system size. An example of this kind of correlations is the so-called random field case (see e.g. [15]), for which  $\gamma = \frac{1}{2}$  and

$$F(y^2) = h^2(N - |y|) \quad (10)$$

where  $N$  is the system size. In the short-ranged case a phase transition in terms of the temperature (or the strength of the disorder) has been found, where the two phases differ in the nature of the distribution of overlaps. A one-step replica-symmetry-breaking (RSB) solution has been found for  $T < T_c$  when using the variational approximation. Both phases

in this approximation were found to be characterized by the trivial wandering exponent  $\nu = \frac{1}{2}$  defined by

$$\overline{\langle x^2(L) \rangle} \propto L^{2\nu} \quad (11)$$

(here the bar represents configurational or thermal average and the brackets refer to averaging over the disorder). This is unlike the exact analytical result at  $d = 2$  which yields  $\nu = \frac{2}{3}$  (superdiffusion) [1, 6], and simulations that found  $\nu > \frac{1}{2}$  for  $d \geq 2$  [17, 18]. There were claims in the literature that the wandering exponent is greater than  $\frac{1}{2}$  for any finite  $d$  in the disordered dominated phase [18]. On the other hand there are claims that have gained more weight recently [19, 11, 20–22], i.e. that the exponent becomes trivial ( $\frac{1}{2}$ ) at an upper critical dimension, which is presumably  $d_c = 5$  (four transverse directions).

For the case of long-ranged correlations the variational approximation for the continuum limit model predicts [8]

$$\nu_{\text{Flory}} = \frac{3}{2(1 + \gamma)} \quad (12)$$

and it is not known if this result is exact at any finite dimension. In this case no phase transition is found as a function of the temperature (or the strength of the disorder) and there is an infinite-step RSB solution (à la Parisi) for the appropriate order parameter, which manifests itself as a continuous (non-delta function) part in the overlap distributions for the walks.

The special case of long-ranged harmonic correlations ( $\gamma = 0$ ) has been solved exactly by Parisi [23]. An inspection of his solution reveals that in that case the free energy is not extensive but grows as  $L^2$  (where  $L$  is the size of the system in the temporal direction). Another exponent which is often referred to in the literature is the exponent characterizing the free energy fluctuations

$$\langle F^2 \rangle - \langle F \rangle^2 \propto L^{2\omega}. \quad (13)$$

The  $\omega$  exponent is related to  $\nu$  through the scaling relation

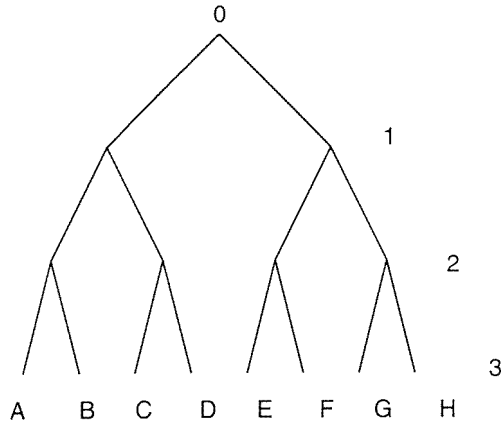
$$2\nu = 1 + \omega. \quad (14)$$

Thus for the harmonic case  $\nu = \frac{3}{2}$  and  $\omega = 2$ . The harmonic case is special in the sense that its solution is replica symmetric.

The case of long-ranged correlations, or in fact any non-zero correlations of the disorder, has not been investigated in the previous treatments of the model on the Cayley tree. The interesting results obtained for the long-ranged case in the continuum limit, and the fact that an exact solution has been found in the case of harmonic correlations, motivated us to carry out an investigation of the case of non-zero ranged spatial correlations on a Cayley tree, both for the case of short- and long-ranged correlations. We obtain some interesting results which will be presented below together with some open questions. A thorough understanding of the directed polymer problem is particularly important because of its connection with the Kardar–Parisi–Zhang equation [24] and with the behaviour of flux lines in high- $T_c$  superconductors [25–28]. There is also a well known mapping from the directed polymer problem to Burgers' turbulence, where the case of long-ranged correlations is of importance [29].

## 2. The tree problem

We consider directed walks on a branch of a Cayley tree of coordination number  $K + 1$  (see figure 1). Each bond branches into  $K$  new bonds in the forward 'time' direction. For



**Figure 1.** A branch of a tree with branching ratio  $K = 2$ . The ultrametric distance between points A and B is 1, between A and C (or D) is 2 and between A and E (or A and F, G, H) is 3.

each site (or alternatively each bond ending at the given site) we choose a random energy  $\epsilon(t, z)$  where  $t$  designates the shell,  $0 \leq t \leq L$ , and  $z$  is a label within the shell that can take  $K^t$  values. The random energies are chosen from a Gaussian distribution satisfying

$$\langle \epsilon(t, z) \rangle = 0 \quad (15)$$

$$\langle \epsilon(t, z) \epsilon(t', z') \rangle = \delta_{t,t'} f(d(z, z')). \quad (16)$$

Here  $f(d)$  is a function to be specified later, and  $d(z, z')$  is a distance among points (sites) belonging to the same shell. Energies at different shells are uncorrelated.

In this paper we choose the following definition for  $d(z, z')$ :

$$d_u(z, z') = \text{number of steps for two walks starting at } z \text{ and } z' \text{ and moving } \textit{backwards} \text{ in time to meet.} \quad (17)$$

This is a hierarchical distance between two points. It also satisfies  $d_u(z, z) = 0$  and  $0 \leq d_u(z, z') \leq t$  within the  $t$ -shell. This distance is also referred to as an ultrametric distance [30] (hence the subscript  $u$ ), since it satisfies a stronger inequality than the ordinary triangular inequality, namely

$$d_u(z, z') \leq \max(d_u(z, y), d_u(y, z')) \quad (18)$$

for any point  $y$  within the shell. This ultrametric distance is very different from an Euclidean distance on lattices characterized by translational invariance in real space, but is sufficient for defining spatial correlations of the disorder within a shell, making the problem amenable to an exact solution.

An alternative choice for a distance which is more suitable for calculating the root-mean-square transverse distance can be defined as follows [19]. Let us label each branch of the tree by  $1, \dots, K$ , which we call directions. These labels are *a priori* arbitrary on a tree, but once the choice is made it remains fixed at each branching point. For a given walk of length  $t$  starting at the origin we denote by  $z_1$  the number of times the walk moves in the 1-direction, by  $z_2$ , the number of times it moves in the 2-direction, etc. We then associate a vector  $(z_1 - t/K, \dots, z_K - t/K)$  with the endpoint of the walk on the  $t$ -shell. We denote this  $K$ -dimensional vector by  $\mathbf{R}(z)$ . Note that there is not a one-to-one correspondence between points  $z$  on the tree and vectors  $\mathbf{R}$  as there are different points which are associated with

the same vector. The transverse distance between two points is defined as

$$d_{tr}(z, z') = \left( \sum_{i=1}^K (R_i(z) - R'_i(z'))^2 \right)^{1/2}. \tag{19}$$

This distance always satisfies the inequality

$$d_{tr}(z, z') \leq \sqrt{2}d_u(z, z') \tag{20}$$

for any two points  $z, z'$ . The advantage of this distance is that in the absence of disorder one has

$$\overline{R} = 0 \tag{21}$$

$$\overline{R^2} = \frac{1}{K} \left( 1 - \frac{1}{K} \right) L \tag{22}$$

where the bar denotes configurational average over all walks of length  $L$ . Thus  $v = \frac{1}{2}$  as is expected for a random walk. In the presence of disorder with spatial correlations, this distance is harder to use in a calculation of the quenched free energy of the model, and further discussion of the use of this distance will be given in a future publication.

### 3. The replica solution

The method we use in this section is a generalization of the method used in appendix 1 of [19] for uncorrelated disorder. To calculate the free energy of the model we use the replica trick

$$-\beta F = \lim_{n \rightarrow 0} \frac{1}{n} \ln \langle Z_L^n \rangle. \tag{23}$$

We can think of  $Z_L^n$  as the partition function of  $n$  different walks of length  $L$  emanating from the origin of a branch of a tree. Adopting Parisi's scheme for RSB in real space, we assume that the following arrangement of the  $n$  walks gives the leading contribution to the free energy in the  $n \rightarrow 0$  limit when  $L$  is large.

(a) The  $n$ -walks stay together for the first  $L(q_1 - q_0)$  steps (where  $q_0 = 0$  and  $0 \leq q_1 \leq 1$ ).

(b) The walks split into  $m_1$  bundles of  $(n/m_1)$  walks each and remain so for a time  $L(q_2 - q_1)$ .

(c) Continuing in this way, in the  $j$ th step the walks split into  $m_j$  groups each comprising of  $(n/m_j)$  walks and remain so for a time  $L(q_{j+1} - q_j) \equiv L\Delta q_j$ .

(d) Finally, at time  $t_M = Lq_M = L$ , the walks split into  $n$  individual walks.

Thus

$$0 = q_0 \leq q_1 \leq \dots \leq q_M = 1 \tag{24}$$

$$1 = m_0 \leq m_1 \leq \dots \leq m_M = n. \tag{25}$$

We also define

$$x(q_j) = n/m_j \tag{26}$$

and thus

$$1 = x(q_M) \leq \dots \leq x(q_j) \dots \leq x(q_0) = n. \tag{27}$$

For large  $L$ ,  $\langle Z_L^n \rangle$  is given by

$$\langle Z_L^n \rangle = \max_{\{q_j\}} \max_{\{m_j\}} \prod_{j=0}^{M-1} (K^{m_j} \langle \exp\{-\beta(n/m_j)(\epsilon_1^{(t)} + \dots + \epsilon_{m_j}^{(t)})\} \rangle)^{L\Delta q_j}. \tag{28}$$

Here  $\epsilon_i^{(t)}$  denotes the energy encountered by a walker belonging to the  $i$ th group at time  $t$ .

The factorization in equation (28) follows from the fact that random energies at different times are uncorrelated. The factor  $K^{m_j}$  is a geometrical degeneracy factor which is entropic in origin. It arises because each of the  $m_j$  groups can choose its next step among  $K$  different possibilities, and each possibility gives rise, as will become clear in the following, to a configuration characterized by the same contribution to the partition function (same Boltzmann weight). One can also associate a combinatorial factor with different possibilities to assign individual walks to bundles when they split, but this turns out to give a total factor of  $n!$ , which becomes 1 in the  $n \rightarrow 0$  limit.

To proceed we use the fact that the random energies are chosen from a Gaussian distribution satisfying equation (16). It then follows that

$$\langle \exp\{-\beta(n/m_j)(\epsilon_1^{(t)} + \dots + \epsilon_{m_j}^{(t)})\} \rangle = \exp\{\frac{1}{2}\beta^2(n/m_j)^2 \langle (\epsilon_1^{(t)} + \dots + \epsilon_{m_j}^{(t)})^2 \rangle\} \quad (29)$$

$$\langle (\epsilon_1^{(t)} + \dots + \epsilon_{m_j}^{(t)})^2 \rangle = m_j f(0) + 2 \sum_{\ell} N_{j,\ell} f(\ell) \quad (30)$$

where  $N_{j,\ell}$  is the number of pairs of distance  $\ell$  at time  $t \in [t_j, t_{j+1}]$  among the  $m_j$  groups of walkers, where  $t_j = Lq_j$ . The coefficients  $N_{j,\ell}$  satisfy

$$m_j + 2 \sum_{\ell} N_{j,\ell} = m_j^2. \quad (31)$$

A careful enumeration of the distribution of distances after  $t$  steps ( $t_j < t < t_{j+1}$ ) reveal that the following identity holds:

$$N_{j,L(q_j - q_k) + \Delta t} = \frac{1}{2} \left( \frac{1}{m_{k-1}} - \frac{1}{m_k} \right) m_j^2 \quad k = 1, \dots, j \quad (32)$$

with  $\Delta t = t - t_j$ . Since we are interested in the limit  $M \rightarrow \infty$ , i.e.  $\Delta q \rightarrow 0$ , we will take  $N_{j,\ell}$  to depend only on  $j$  and omit  $\Delta t$  in equation (32). In deriving equation (32) we used the fact that the groups of walkers split at each time  $t_j = Lq_j$  according to the procedure described at the beginning of the section, as well as the definition of the hierarchical (ultrametric) distance. Substituting the result (32) in equation (30) we find

$$(n/m_j)^2 \langle (\epsilon_1^{(t)} + \dots + \epsilon_{m_j}^{(t)})^2 \rangle = nx(q_j) f(0) + n \sum_{k=1}^j (x(q_{k-1}) - x(q_k)) f(L(q_j - q_k)). \quad (33)$$

In the limit  $n \rightarrow 0$  the inequalities (25) and (27) are inverted and hence

$$0 = x(q_0) \leq \dots \leq x(q_j) \dots \leq x(q_M) = 1. \quad (34)$$

Using expression (33) in equation (29) and subsequently in the formula for  $\langle Z_L^n \rangle$ , equation (28), we obtain

$$\begin{aligned} \frac{1}{nL} \ln \langle Z_L^n \rangle &= \sum_{j=0}^{M-1} \frac{\Delta q_j}{x(q_j)} \ln K + \frac{1}{2} \beta^2 \sum_{j=0}^{M-1} \Delta q_j x(q_j) f(0) \\ &\quad - \frac{1}{2} \beta^2 \sum_{j=0}^{M-1} \Delta q_j \sum_{k=1}^j \Delta q_k \left( \frac{x(q_k) - x(q_{k-1})}{\Delta q_k} \right) f(L(q_j - q_k)) \end{aligned} \quad (35)$$

where an extremum over  $x(q_j)$  has to be taken. In the limit of large  $M$ ,  $q$  becomes a continuous variable in the interval  $[0,1]$ , and  $x(q)$  becomes a function on that interval satisfying  $0 \leq x(q) \leq 1$ . The summations in equation (35) become integrals and we have

$$\frac{-\beta F}{L} = \int_0^1 \frac{dq}{x(q)} \ln K + \frac{1}{2} \beta^2 \int_0^1 dq x(q) f(0) - \frac{1}{2} \beta^2 \int_0^1 dq \int_0^q dp x'(p) f(L(q-p)) \quad (36)$$

where  $x'(p)$  stands for  $dx/dp$ . This expression can be further simplified: first we perform the  $p$  integration by parts using the fact that  $x(0) = 0$ . Next, in the second term which remains a double integral we change the order of integration

$$\int_0^1 dq \int_0^q dp = \int_0^1 dp \int_p^1 dq \tag{37}$$

leading finally to a single integral. These steps yield

$$\frac{-\beta F}{L} = \int_0^1 \frac{dq}{x(q)} \ln K + \frac{1}{2}\beta^2 \int_0^1 dq x(q)[f(0) - f(L(1 - q))] \tag{38}$$

where  $x(q)$  is to be determined by extremizing this expression. Equation (38) is the main result of this section.

Before we end this section let us mention a simple generalization of the problem. Equation (16) can be replaced by a more general form

$$\langle \epsilon(t, z)\epsilon(t', z') \rangle = \delta_{t,t'} f(t, d(z, z')) \tag{39}$$

so the spatial correlation can vary with time (but energies at different times are still uncorrelated). The possibility of a time-dependent width of the distribution of disorder in the uncorrelated case has been considered in [13]. Using equation (39), we can repeat all the steps leading to equation (36), and we find that it is replaced by

$$\begin{aligned} \frac{-\beta F}{L} = & \int_0^1 \frac{dq}{x(q)} \ln K + \frac{1}{2}\beta^2 \int_0^1 dq x(q) f(Lq, 0) \\ & - \frac{1}{2}\beta^2 \int_0^1 dq \int_0^q dp x'(p) f(Lq, L(q - p)). \end{aligned} \tag{40}$$

It is easy to pursue this more general case by the same method presented in the next section, but we will not consider it further in this paper.

#### 4. Solution for the free energy and overlap distribution

In order to extremize the expression for the free energy (38) derived in the last section, we take a functional derivative with respect to  $x(q)$  to obtain

$$-\frac{\ln K}{x^2(q)} + \frac{1}{2}\beta^2(f(0) - f(L(1 - q))) = 0 \tag{41}$$

and hence

$$x(q) = \frac{T\sqrt{2 \ln K}}{\sqrt{f(0) - f(L(1 - q))}}. \tag{42}$$

This solution is valid for the range of  $q$  for which the inequality  $0 \leq x(q) \leq 1$  holds. Otherwise one has to choose  $x(q)$  at its maximal (or minimal) allowed values.

Before we consider some concrete candidates for the spatial correlation function, we make a comment regarding possible admissible correlations. The Schwarz inequality requires that the function  $f(y)$  defined in equation (16) satisfy  $|f(y)| \leq f(0)$ . More generally for the  $k$ th level the covariance matrix

$$C_{z,z'} = \langle \epsilon(k, z)\epsilon(k, z') \rangle = f(d(z, z')) \tag{43}$$

must be positive semidefinite, which means that all its eigenvalues are non-negative. In the  $k$ th shell the hierarchical distance takes values from 1 to  $k$ . Writing down the covariance matrix for the  $k$ th shell we verified that a sufficient condition for an admissible correlation



of the disorder is a function  $f(y)$  which is always non-negative and is a monotonically decreasing function of the hierarchical distance  $y$ . For simplicity we will limit the discussion for the case  $K = 2$ , but it can be generalized to any value of  $K$ . First we present as an example the eigenvalues of the covariance matrix in the three-dimensional shell of the Bethe lattice (BL)

$$\begin{aligned} f(0) - f(1) & \quad (\text{multiplicity } 4) \\ f(0) + f(1) - 2f(2) & \quad (\text{multiplicity } 2) \\ f(0) + f(1) + 2f(2) - 4f(3) \\ f(0) + f(1) + 2f(2) + 4f(3) \end{aligned}$$

and these are all non-negative if  $f(0) > f(1) > f(2) > f(3) \geq 0$ . In general we now obtain by induction all the eigenvalues of the covariance matrix of the  $(k + 1)$ th shell once we know the eigenvalues of the matrix of the  $k$ th shell. For  $k = 1$  the covariant matrix is given by

$$\begin{pmatrix} f(0) & f(1) \\ f(1) & f(0) \end{pmatrix} \quad (44)$$

which has eigenvectors  $(1, -1)$  and  $(1, 1)$  and corresponding eigenvalues  $f(0) - f(1)$  and  $f(0) + f(1)$ . For  $k = 2$  the covariant matrix is a  $4 \times 4$  matrix

$$C(2) = \begin{pmatrix} C(1) & D(1) \\ D(1) & C(1) \end{pmatrix} \quad (45)$$

where  $C(1)$  is the  $k = 1$  covariance matrix and  $D(1)$  is a  $2 \times 2$  matrix, all the elements of which are equal to  $f(2)$ . The eigenvectors of  $C(2)$  are  $(1, -1, 0, 0)$  and  $(0, 0, 1, -1)$  with the corresponding eigenvalue  $f(0) - f(1)$  of multiplicity 2. In addition there are two new eigenvectors  $(1, 1, -1, -1)$  and  $(1, 1, 1, 1)$  with eigenvalues  $f(0) + f(1) - 2f(2)$  and  $f(0) + f(1) + 2f(2)$ . In the  $k$ th level there are always  $k + 1$  distinct eigenvalues, the  $(k + 1)$ th of which is always given by

$$\lambda_{k+1}(k) = f(0) + f(1) + 2f(2) + \dots + 2^{k-1}f(k). \quad (46)$$

At level  $(k + 1)$  the first  $k$  eigenvalues remain the same as level  $k$  first  $k$  eigenvalues, but with double multiplicity. This is because the corresponding eigenvectors are just obtained from the previous eigenvectors by adding zeros at the beginning or at the end. But there are two new eigenvectors  $(1, \dots, 1, -1, \dots, -1)$  and  $(1, \dots, 1)$  with corresponding eigenvalues which are given by

$$\begin{aligned} \lambda_{k+1}(k+1) &= \lambda_{k+1}(k) - 2^k f(k+1) \\ \lambda_{k+2}(k+1) &= \lambda_{k+1}(k) + 2^k f(k+1). \end{aligned} \quad (47)$$

It is thus straightforward to check that the positivity and (decreasing) monotonicity conditions mentioned above yield only non-negative eigenvalues. We now turn to some concrete examples.

(i) *The case of short-ranged spatial correlations.* We first consider the function

$$f(y) = \frac{g}{(a_0 + y)^\lambda} \quad (48)$$

with  $\lambda > 0$ . In this case equation (42) becomes

$$x(q) = \frac{T\sqrt{2\ln K/g}}{\{a_0^{-\lambda} - [a_0 + L(1-q)]^{-\lambda}\}^{1/2}} \quad (49)$$

and in the limit of large  $L$  the solution for  $x(q)$ ,  $0 < q < 1$  becomes

$$x(q) = \begin{cases} \frac{T}{T_c} & T \leq T_c \\ 1 & T \geq T_c \end{cases} \tag{50}$$

together with  $x(0) = 0$  and  $x(1) = 1$ . Here  $T_c$  is given by

$$T_c = \sqrt{\frac{f(0)}{2 \ln K}}. \tag{51}$$

For  $T \leq T_c$ , deviations of  $x(q)$  from the form given in expression (50) are of  $O(L^{-\lambda})$ , except when  $1 - q \sim O(1/L)$ , and  $x(q_c)$  becomes equal to 1 for  $q_c = 1 - O(1/L)$ .

The solution found above is the same as the solution found in [13], for zero-ranged correlations. Thus we see that the case of short-ranged correlations is characterized by the same overlap function and free energy as the zero-ranged case. The overlap distribution function is given in terms of the solution for  $x(q)$  by [30]

$$P(q) = \frac{dx(q)}{dq} \tag{52}$$

and hence

$$P(q) = \begin{cases} \frac{T}{T_c} \delta(q) + \left(1 - \frac{T}{T_c}\right) \delta(q - 1) & T \leq T_c \\ \delta(q) & T \geq T_c \end{cases} \tag{53}$$

as discussed in the introduction. The free energy is obtained by substituting the expression for  $x(q)$  in equation (38) and we obtain:

$$-F/L = \begin{cases} T_c \ln K + f(0)/(2T_c) = \sqrt{2f(0) \ln K} & T \leq T_c \\ T \ln K + f(0)/(2T) & T \geq T_c \end{cases} \tag{54}$$

where we have dropped corrections on the rhs which vanish as  $L \rightarrow \infty$ . Note that on the BL with the hierarchical distance we did not find any substantial change as  $\lambda$  crosses the value 1 as has been found in the continuum limit.

(ii) *The case of long-ranged correlations.* In this case we choose for the function governing the disorder correlation

$$f(y) = a_0 - gy^\alpha \tag{55}$$

with  $\alpha > 0$  and  $g > 0$ . We take  $a_0$  to be very large, so that it satisfies

$$T \ll \sqrt{\frac{a_0}{2 \ln K}} \tag{56}$$

for the range of temperatures we are interested in. Furthermore, if  $L$  is such that it satisfies

$$L < \left(\frac{a_0}{g}\right)^{1/\alpha} \tag{57}$$

then  $f(y)$  is positive for the entire range of allowed distances and since it is a monotonically decreasing function of  $y$  it is a bona fide correlation. The region of physical interest is  $a_0 \rightarrow \infty$  with  $L$  large but still satisfying the condition (57). Eventually though, if one is interested in the true thermodynamic limit  $L \rightarrow \infty$  one has to consider the case  $L > (a_0/g)^{1/\alpha}$ . In that case we must define

$$f(y) = \begin{cases} a_0 - gy^\alpha & 0 \leq y \leq (a_0/g)^{1/\alpha} \\ 0 & (a_0/g)^{1/\alpha} \leq y \leq L \end{cases} \tag{58}$$

for  $f(y)$  to be a proper correlation function. We will comment about the thermodynamic limit later.

For now, starting with  $f(y)$  given by equation (55) with the condition (57) being satisfied, the solution for  $x(q)$  (see equation (42)) becomes

$$x(q) = \begin{cases} \frac{T\sqrt{2\ln K}}{\sqrt{gL^\alpha(1-q)^\alpha}} & 0 \leq q \leq q_c \\ 1 & q_c \leq q \leq 1 \end{cases} \quad (59)$$

with

$$q_c = 1 - \frac{1}{L} \left( \frac{T^2 2 \ln K}{g} \right)^{1/\alpha}. \quad (60)$$

We can identify an  $L$ -dependent crossover temperature (at which  $q_c = 0$ ) given by

$$T_c(L) = \left( \frac{g}{2 \ln K} \right)^{1/2} L^{\alpha/2}. \quad (61)$$

For  $T$  fixed, as  $L$  becomes large,  $T_c(L)$  grows with  $L$  and thus we find ourselves constantly in the low-temperature phase of the model, characterized by  $q_c > 0$ . (If on the other hand  $T > T_c(L)$  then  $x(q) = 1$  for any  $q > 0$ .) Assuming that  $L$  is large enough so  $T \ll T_c(L)$  (but with condition (57) still satisfied), we find by substituting the expression for  $x(q)$  in the formula for the free energy, equation (38)

$$-\frac{\beta F}{L} = \frac{T_c(L) \ln K}{T} \int_0^1 dq (1-q)^{\alpha/2} + \frac{T}{T_c(L)} \frac{gL^\alpha}{2T^2} \int_0^1 dq (1-q)^{\alpha/2} + O(L^{-1}) \quad (62)$$

which can be simplified to give

$$-F = \frac{\sqrt{2g \ln K}}{1 + \alpha/2} L^{1+\alpha/2} \quad (63)$$

where we dropped constant terms. We see that the free energy is not extensive for this range of  $L$  values, but rather proportional to  $L^{1+\alpha/2}$ . It is also temperature independent as it is in the low-temperature phase of the short-ranged case.

Let us now consider the distribution of overlaps. Using equation (52) we find

$$P(q) = \begin{cases} \frac{T}{T_c(L)} \delta(q) + \frac{\alpha}{2} \frac{T}{T_c(L)} \frac{1}{(1-q)^{1+\alpha/2}} & 0 \leq q \leq q_c \\ 0 & q_c \leq q \leq 1 \end{cases} \quad (64)$$

in the limit of large  $L$  this expression becomes simply

$$P(q) = \delta(1-q). \quad (65)$$

This becomes obvious by going back to the expression for  $x(q)$  which in the limit of large  $L$  become  $x(q) = 0$  for  $0 \leq q < 1$  and  $x(1) = 1$ , which can also be expressed as

$$q(x) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x \leq 1. \end{cases} \quad (66)$$

Thus in the limit of large  $L$  the solution becomes replica symmetric for any fixed temperature.

Before we close this section we show that in the true thermodynamic limit the free energy becomes extensive. To achieve this we must allow  $L > (a_0/g)^{1/\alpha}$  and use the correlation

function defined by eq.(58). We still demand that  $a_0$  be very large so  $T \ll \sqrt{a_0/(2 \ln K)}$  is always satisfied. In that case we find for  $x(q)$ :

$$x(q) = \begin{cases} T/T_c(a_0) & 0 \leq q \leq q_{c1} \\ \frac{T\sqrt{2 \ln K}}{\sqrt{gL^\alpha(1-q)^\alpha}} & q_{c1} \leq q \leq q_{c2} \\ 1 & q_{c2} \leq q \leq 1 \end{cases} \quad (67)$$

where

$$T_c(a_0) = \left( \frac{a_0}{2 \ln K} \right)^{1/2} \quad (68)$$

$$q_{c1} = 1 - \frac{1}{L} \left( \frac{a_0}{g} \right)^{1/\alpha} \quad (69)$$

$$q_{c2} = 1 - \frac{1}{L} \left( \frac{T^2 2 \ln K}{g} \right)^{1/\alpha}. \quad (70)$$

In the limit  $L \rightarrow \infty$  one finds

$$x(q) = T/T_c(a_0) \quad 0 < q < 1 \quad (71)$$

together with  $x(0) = 0$  and  $x(1) = 1$ . In the limit  $T \ll T_c(a_0)$  we see that practically  $x(q) = 0$  for  $0 \leq q < 1$  and  $x(1) = 1$  which amounts to  $P(q) = \delta(1 - q)$  as before. The free energy though is given by

$$-F/L = \sqrt{2a_0 \ln K} \quad (72)$$

and it is thus an extensive function of  $L$  as it should be in the thermodynamic limit.

## 5. Summary and discussion

In this paper we have considered the case of directed polymers on a Cayley tree in the presence of correlated disorder. We have used the ultrametric hierarchical distance to introduce distance within each shell, and this distance is simple enough to enable us to solve the model exactly under the assumption of a hierarchical Parisi-type solution.

We have found two different types of behaviour depending on the range of the disorder correlations. In the case of short-range correlations the solution behaves like the non-correlated case: there is a phase transition at a finite temperature and the two phases differ by the temperature dependence of the free energy and by the expression for the overlap distribution which is non-trivial in the low-temperature phase. In the case of long-range correlations there is no phase transition as a function of temperature (strictly speaking the transition temperature  $T_c(a_0)$  can be made as large as we please by choosing  $a_0$  to be large enough). This is similar to the behaviour that has been found in the continuum limit [8]. However, we have identified an  $L$ -dependent crossover temperature which plays a role for a finite-size system. We have also found that in the large  $L$  limit the solution becomes replica-symmetric but with the overlap distribution peaked at  $q = 1$  at any temperature, which is the case in the short-ranged case only at  $T = 0$ . This result is reminiscent of Parisi's solution [23] for the case of harmonic correlations in the continuum limit where no RSB has been found. This is in contrast to results in the continuum limit for the non-harmonic case [8] where the variational approximation yielded an infinite-step RSB for the long-ranged case (but also no phase transition). Another feature we have found in the long-ranged case, which is similar to Parisi's result [23], is the non-extensivity of the free

energy in terms of  $L$  over a large range of  $L$ -values. Eventually as  $L \rightarrow \infty$ , the free energy become extensive. The exponent  $1 + \alpha/2$  may be related to the exponent  $\omega$  which characterizes the free energy fluctuations, see equation (13), but this can be established only after carrying out a calculation of the free energy fluctuations on the Cayley tree.

We should emphasize, that because of the non-Euclidean nature of the hierarchical distance on a tree, we could not establish a relation between the exponents  $\lambda$  in equation (48) or  $\alpha$  in equation (55) to the exponent  $\gamma$  defined in equations (8) and (9) which characterizes the range of disorder correlations on ordinary lattices embedded in Euclidean space. Related to this is the fact that the separation between short- and long-range correlations occurs for  $\gamma = 2$  for ordinary lattices (see equation (8) and [8]) whereas we find the separation to occur at  $\lambda = 0$  (or  $\alpha = 0$ ) for the tree problem.

There are various possibilities to extend this work further. One is to consider the transverse distance defined in section 2, and attempt to solve the model, including the behaviour of the root-mean-square transverse distance characterized by an exponent  $\nu$ .

Also, we have only considered the case of a Gaussian distribution of the disorder. Other distributions are possible, some better suited for calculating  $1/d$  corrections (like the exponential distribution) [19]. One can attempt to obtain these corrections for the case of long-ranged correlations of the disorder.

## Acknowledgments

I thank T Blum for a useful discussion and comments. I thank the referee for useful comments and suggestions. This work is partially supported by the US Department of Energy, grant no DE-G02-98ER45686.

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